Exactly solvable model of continuous stationary 1/*f* noise

James P. Gleeson*
Applied Mathematics, University College Cork, Cork, Ireland
(Received 16 December 2004; published 18 July 2005)

An exactly solvable model generating a continuous random process with a $1/f$ power spectrum is presented. Examples of such processes include the angular (phase) speed of trajectories near stable equilibrium points in two-dimensional dynamical systems perturbed by colored Gaussian noise. An exact formula giving the correlation function of the $1/f$ noise in terms of the correlation of the perturbing colored noises is derived, and used to show that the $1/f$ spectrum is found in a wide variety of cases. The $1/f$ noise is non-Gaussian, as demonstrated by calculating its one-time probability distribution function. Numerical simulations confirm and extend the theoretical results.

DOI: 10.1103/PhysRevE.72.011106 PACS number(s): 05.40.−a, 02.50.−r

I. INTRODUCTION

The characterization "$1/f$ noise" is applied to the wide variety of time-varying random processes whose power spectra have extended regions of $f^{-\delta}$ scaling, with $\delta \approx 1$. The terms "flicker noise" and "pink noise" are also commonly used. Such noise appears in a remarkably diverse range of physical phenomena, from semiconductor device noise to sea level time series [1]. This range has led to many efforts to model the origin of such noise in a universal fashion, seeking a single underlying cause or mechanism generating all $1/f$ noises. The website [2] and review article [3] provide a comprehensive bibliography of $1/f$ noise and models thereof; here we list only a few selected examples. Models built upon the ideas of self-organized criticality [4] have been shown to generate $1/f$ noises in certain parameter regimes; note that the time variable in such systems is typically discrete [5,6]. An approximation to continuous-time $1/f$ noise may be generated by a superposition of a large number of Ornstein-Uhlenbeck processes, with corner frequencies (relaxation times) which are distributed in an appropriate fashion [7]. The resulting process then has an approximate $1/f$ spectral range, with cutoffs at both low and high frequencies. Below the lower cutoff the spectrum reaches a constant finite value, while above the upper cutoff it decays as f^{-2}, typical of Ornstein-Uhlenbeck processes. The possibility of such a superposition of processes arising from linear diffusion and transport processes with white-noise sources has been demonstrated in [7].

Analytically tractable models yielding exact $1/f$ processes are rare, but provide very useful examples and tests for more general approximation schemes. Efficient numerical methods for generating $1/f$ noise are useful in electronic circuit simulators, and exactly solvable models can suggest new methods for generating such noise with precisely determined characteristics. An analytically solvable model where the signal consists of a sequence of δ-function pulses was presented in [8]; such models arise in, for example, flows of granular materials [9]. The (discrete) autoregressive process for the recurrence times of the pulses is forced by white noise, and can produce a $1/f$ spectrum in a wide, but finite, frequency range. To our knowledge no model of *continuous-time* $1/f$ noise that is exactly solvable (in the sense of giving an explicit formula for the correlation function or spectrum of the process while making no approximations) has previously appeared. The main result of this paper is a proof that the stationary (and continuous) random process $v(t)=d\theta/dt$ has a $1/f$ spectrum, where $\theta(t)$ is the phase angle defined by

$$\theta(t) = \tan^{-1}\frac{y(t)}{x(t)}, \tag{1}$$

with $x(t)$ and $y(t)$ being identical, independent, zero-mean Gaussian colored noises satisfying condition (16) below. We note that such a phase angle arises naturally in many situations of physical interest, and indeed the statistics of phase is an important topic in many fields; see for instance [10,11]. An example yielding precisely the random process (1) is given by the rate-of-strain matrix for a simple model of two-dimensional incompressible fluid turbulence [12]:

$$S = \begin{pmatrix} a(t) & b(t) \\ b(t) & -a(t) \end{pmatrix}, \tag{2}$$

where the independent components $a(t)$ and $b(t)$ are zero-mean Gaussian colored-noise processes. The eigenvector of S with positive eigenvalue defines the principal axis of strain, along which infinitesimal line elements in the fluid become aligned [13]. This eigenvector makes an angle θ with the x axis, where

$$\theta(t) = \frac{1}{2}\tan^{-1}\frac{b(t)}{a(t)} \tag{3}$$

and the speed of rotation $d\theta/dt$ then constitutes an example of a $1/f$ process as detailed below. Another example is provided by a dynamical disorder model for the logarithmic stock price of an equity, as compared with market data [14,15]. Our main example, which provides the notation we adopt for the remainder of the paper, concerns fluctuations of a two-dimensional dynamical system about a stable equilibrium point in the presence of colored noise. Linearizing the dynamical system about the equilibrium point (shifted to the

*Email address: j.gleeson@ucc.ie

origin), the equations of motion for the trajectory $(x(t),y(t))$ are

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = J\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \eta \\ \xi \end{pmatrix}, \tag{4}$$

where J is the 2×2 Jacobian matrix evaluated at the origin, and $\eta(t)$ and $\xi(t)$ are independent zero-mean Gaussian colored-noise processes [satisfying the mild condition (17)]. The equilibrium is stable to small disturbances provided that both eigenvalues of J have negative real parts. In the absence of noise, the probability distribution function (PDF) of the trajectories is a δ-spike at the origin; the influence of noise is to spread this spike into a finite-sized PDF cloud around the origin. For example, if the Jacobian has the simple form

$$J = \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix} \tag{5}$$

for $\alpha > 0$, then the PDF cloud is circularly symmetric about the origin. As the trajectories have negligible probability of passing exactly through the origin, the phase angle

$$\theta(t) = \tan^{-1}\frac{y(t)}{x(t)} \tag{6}$$

is well defined, and corresponds to the angular variable in a polar coordinate description of the motion. The phase speed

$$v(t) = \theta' = \frac{xy' - yx'}{x^2 + y^2} \tag{7}$$

is then a stationary random variable, and our claim is that the spectrum of $v(t)$ scales as $1/f$ for high frequencies f, provided that η and ξ are colored noises.

This result is rather remarkable, because it holds when η and ξ are colored noises with almost any spectral shape (subject to the constraint described below), but is quite different from the case where the additive noise terms in (4) are white. Taking the Jacobian of Eq. (5) for example, the phase dynamics of (4) in polar coordinates (with $r=\sqrt{x^2+y^2}$) are given by

$$r\theta' = -x'\sin\theta + y'\cos\theta = -\eta(t)\sin\theta + \xi(t)\cos\theta. \tag{8}$$

When η and ξ are white noises, the vanishing correlation time means that the right hand side of Eq. (8) may be replaced by a single white noise (see, for example, [16])

$$r\theta' = \zeta(t) \tag{9}$$

and so the θ' process is also a white noise. In Sec. III we prove that if the noise processes η and ξ are colored (with spectra decaying faster than f^{-1} as $f\to\infty$) then the phase speed θ' has a $1/f$ spectrum.

Note that the inverse tangent function which defines the phase angle in (6) is often limited in range to $[0,2\pi]$. For instance, as $\theta(t)$ moves smoothly through 2π (with $x>0$ and y passing through zero from below), the inverse tangent function jumps from 2π to zero. We stress that the effects of such artificial jumps are not seen in the phase speed as defined in (7). All the work presented here is based upon this definition of the phase speed, and so we are effectively treating the angle $\theta(t)$ as having infinite range or, equivalently, using a continuous version of the inverse tangent function in (6).

The remainder of this paper is organized as follows. After introducing some notation and spectral representations in Sec. II, we derive in Sec. III an exact formula giving the correlation function $R_v(t)$ of the phase speed $v=\theta'$ in terms of the correlation function $R(t)$ of x and y. We examine the small-t behavior of $R_v(t)$ and prove that the corresponding spectrum has a $1/f$ tail of infinite extent. The phase speed $v(t)$ is not a Gaussian process [even though $x(t)$ and $y(t)$ are Gaussian]; this is shown explicitly by calculating the PDF of v. Finally, numerical examples of $1/f$ spectra of the type described here are given in Sec. IV, with numerical experiments extending the theoretical results to show that they are quite robust.

II. NOISE SPECTRA AND CONSTRAINTS

Stationary zero-mean random processes are often characterized by their correlation functions, for example,

$$\langle x(t_1)x(t_1+t)\rangle = R(t), \tag{10}$$

or equivalently by their spectrum

$$S(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ift}R(t)dt. \tag{11}$$

Here angle brackets denote averaging over an ensemble of realizations. Both $R(t)$ and $S(f)$ are real valued, even functions of their arguments. The correlation $R(t)$ may be expressed as the inverse of the Fourier transform (11); we note this integral form shows that $R(t)$ exists only if $S(f)$ is integrable as the frequency f limits to zero—in other words, a stationary random process must have a spectrum that is integrable at low frequencies. In particular, a truly stationary $1/f$ process can exist only if its spectrum has some low-frequency cutoff for the $1/f$ behavior. Conversely, if the $1/f$ scaling continues to arbitrarily low frequencies, then the process cannot be stationary.

In the linearized dynamical system proposed in Eq. (4), the trajectory components $x(t)$ and $y(t)$ are forced by noise terms $\eta(t)$ and $\xi(t)$. The spectrum $S(f)$ of the stationary process $x(t)$ satisfying the first equation of (4),

$$x' = -\alpha x + \eta(t), \tag{12}$$

is easily shown to be given by

$$S(f) = \frac{1}{\alpha^2 + f^2}S_\eta(f), \tag{13}$$

where S_η is the spectrum of the forcing noise process $\eta(t)$. Thus $x(t)$ is a "filtered" version of the noise $\eta(t)$, with a faster-decaying spectrum at high frequencies. The filtering is a smoothing process, so the correlation $R(t)$ of $x(t)$ is better behaved (having, for example, higher derivatives existing) at $t=0$ than the corresponding correlation $R_\eta(t)$ of the noise $\eta(t)$. For instance, if $\eta(t)$ is an Ornstein-Uhlenbeck (OU) process with correlation function

$$R_\eta(t) = \gamma e^{-|t|/\tau} \tag{14}$$

and spectrum

$$S_\eta(f) = \frac{\gamma\tau}{\pi(1+\tau^2 f^2)}, \tag{15}$$

then the derivative $R'_\eta(0)$ at $t=0$ does not exist [equivalently, the integral $\int_0^\infty f S_\eta(f) df$ does not exist]. However, the corresponding filtered trajectory component $x(t)$ as defined in (12) has spectrum given by (13), and the additional smoothness means the correlation function $R(t)$ of $x(t)$ has the desirable properties $R'(0)=0$ and $|R''(0)|<\infty$. The result we derive below depends critically upon the smoothness of the correlation function of $x(t)$ and $y(t)$; in particular, we require that $R'(0)=0$ and $|R''(0)|<\infty$. In terms of the spectrum $S(f)$ of x and y, this requires that

$$\left| \int_{-\infty}^{\infty} f^2 S(f) df \right| < \infty \tag{16}$$

or, using (13) to express the condition in terms of the spectrum S_η of the noise processes,

$$\left| \int_{-\infty}^{\infty} \frac{f^2}{\alpha^2+f^2} S_\eta(f) df \right| < \infty. \tag{17}$$

It is clear that a white noise with its constant spectrum is not therefore admissible as a candidate for the $\eta(t)$ noise process; however, an OU noise spectrum (15) for S_η satisfies (17) and so is admissible. Indeed the high-frequency components of the integral (17) imply that the noise processes $\eta(t)$ and $\xi(t)$ need only have spectra which fall off faster than f^{-1} as $f\to\infty$: this includes a very wide class of common colored-noise processes.

III. PHASE SPEED CORRELATION FUNCTION

The phase speed is defined in terms of the Gaussian processes $x(t)$ and $y(t)$ as

$$v(t) = \frac{d\theta}{dt} = \frac{xy' - yx'}{x^2+y^2}. \tag{18}$$

In this section we assume that $x(t)$ and $y(t)$ have the same correlation function $R(t)$:

$$R(t) = \langle x(t_1)x(t_1+t)\rangle = \langle y(t_1)y(t_1+t)\rangle. \tag{19}$$

We further assume that $R(t)$ is sufficiently smooth so that $R'(0)=0$ and $|R''(0)|<\infty$; as discussed in sec. II this is assured if the forcing noise processes $\eta(t)$ and $\xi(t)$ have spectra obeying the mild constraint (17). Under these assumptions we derive an exact formula for the correlation function of the phase speed,

$$R_v(t) = \langle v(t_1)v(t_1+t)\rangle. \tag{20}$$

Note from (18) that x and y may be multiplied by an arbitrary constant factor without affecting $v(t)$; thus their variance $R(0)$ may be set to unity for convenience. As x and y are independent Gaussian processes, the correlation function of the phase speed may be written as an eight-dimensional Gaussian integral:

$$\begin{aligned} R_v(t) &= \left\langle \left(\frac{x_1y_1' - y_1x_1'}{x_1^2+y_1^2}\right)\left(\frac{x_2y_2' - y_2x_2'}{x_2^2+y_2^2}\right)\right\rangle \\ &= \int \cdots \int \frac{x_1y_1' - y_1x_1'}{x_1^2+y_1^2}\frac{x_2y_2' - y_2x_2'}{x_2^2+y_2^2} \\ &\quad \times P_{12}(x_1,x_1',x_2,x_2')P_{12}(y_1,y_1',y_2,y_2'). \end{aligned} \tag{21}$$

Here the eightfold integrals are over x, y, x', and y' evaluated at each of the two times t_1 (with subscript 1) and $t_2=t_1+t$ (subscript 2). The random processes x and y are independent of each other, but the joint probability function of x and its derivative at different times is found from the Gaussian joint PDF

$$P_{12}(\mathbf{w}) = \frac{1}{(2\pi)^2}(\det\mathbf{A})^{1/2}\exp\left[-\frac{1}{2}\mathbf{w}^T\mathbf{A}\mathbf{w}\right] \tag{22}$$

for the vector of arguments, $\mathbf{w}=(x_1,x_1',x_2,x_2')$, with a similar expression for y. The matrix $\mathbf{A}$ is defined by

$$A_{ij}^{-1} = \langle w_i w_j\rangle \tag{23}$$

and when written out in full:

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & R(t) & R'(t) \\ 0 & -R''(0) & -R'(t) & -R''(t) \\ R(t) & -R'(t) & 1 & 0 \\ R'(t) & -R''(t) & 0 & -R''(0) \end{pmatrix}. \tag{24}$$

Note the zero entries here, following from the assumption $R'(0)=0$ as discussed in Sec. II. Using Eqs. (22) and (24) in Eq. (21) enables us to calculate the correlation function $R_v(t)$. We briefly describe here the steps in the calculation of the multidimensional integrals. First, the four integrals over the variables x_1', y_1', x_2', y_2' are performed—the dependence of the integrand on each of these variables is linear, so the Gaussian integrals may be calculated straightforwardly. This yields

$$\begin{aligned} R_v(t) &= \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dy_1 \\ &\quad \times \int_{-\infty}^{\infty} dy_2 e^{-c_1(x_1^2+x_2^2+y_1^2+y_2^2)+c_2(x_1x_2+y_1y_2)} \\ &\quad \times \frac{c_3(x_1x_2+y_1y_2)+c_4(x_2y_1-x_1y_2)^2}{(x_1^2+y_1^2)(x_2^2+y_2^2)}, \end{aligned} \tag{25}$$

where the time-dependent coefficients are

$$c_1 = \frac{1}{2(1-R(t)^2)},$$

$$c_2 = \frac{R(t)}{1-R(t)^2},$$

$$c_3 = \frac{R(t)^2R''(t) - R''(t) - R(t)R'(t)^2}{4\pi^2[1-R(t)^2]^2},$$

$$c_4 = \frac{R'(t)^2}{4\pi^2[1-R(t)^2]^3}. \tag{26}$$

For the remaining four variables, we transform to polar coordinates in the (x_1, y_1) and (x_2, y_2) planes:

$$x_1 = q\cos\psi, \quad y_1 = q\sin\psi, \quad x_2 = \rho\cos\phi, \quad y_2 = \rho\sin\phi, \tag{27}$$

to obtain

$$R_v(t) = \int_0^\infty dq \int_0^\infty d\rho \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi\, e^{-c_1(q^2+\rho^2)+c_2 q\rho\cos(\psi-\phi)} \times [c_3\cos(\psi-\phi) + c_4 q\rho\sin^2(\psi-\phi)]. \tag{28}$$

The ϕ integral is then trivial, and by integrating over ψ we obtain

$$R_v(t) = 4\pi^2\left(\frac{c_2c_3+c_4}{c_2}\right)\int_0^\infty dq \int_0^\infty d\rho\, e^{-c_1(q^2+\rho^2)} I_1(c_2 q\rho), \tag{29}$$

where I_1 is the modified Bessel function of the first kind. The final integrations yield

$$R_v(t) = 2\pi^2\left(\frac{c_2c_3+c_4}{c_2^2}\right)\ln\left(\frac{4c_1^2}{4c_1^2-c_2^2}\right), \tag{30}$$

and restoring the coefficients from Eq. (26) gives our main result

$$R_v(t) = \frac{1}{2R(t)^2}\{R(t)R''(t) - R'(t)^2\}\ln[1-R(t)^2]. \tag{31}$$

This formula gives the correlation function $R_v(t)$ of the phase speed $v(t)$ in terms of the known correlation function $R(t)$ of $x(t)$ and $y(t)$. The spectrum $S_v(f)$ of the phase speed is given by the Fourier transform of (31); this is examined in detail in the following subsection.

A. Small-t expansion of correlation function and $1/f$ spectrum

Provided that the correlation function $R(t)$ of x and y is sufficiently smooth that $R''(0)$ exists and $R'(0)=0$ [this is guaranteed if the noise sources are sufficiently colored to obey the constraint (17)], a small-t expansion of the phase speed correlation $R_v(t)$ from Eq. (31) yields

$$R_v(t) \sim -\frac{1}{2\tau_0^2}\ln\left(\frac{t^2}{\tau_0^2}\right) + O(t^2\ln t) \quad \text{as } t\to 0. \tag{32}$$

Here the time scale τ_0 is defined by the initial radius of curvature of $R(t)$:

$$\tau_0 \equiv \frac{1}{\sqrt{-R''(0)}}. \tag{33}$$

The logarithmic divergence of $R_v(t)$ as $t\to 0$ means that the variance $R_v(0)$ of $v(t)$ does not exist—this can also be seen from the explicit form of the PDF of v, as shown below. Moreover, the logarithmic form of (32) is independent of the detailed form of the correlation function of x and y, except for its dependence on τ_0.

The phase speed spectrum is defined as

$$S_v(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ift}R_v(t)dt, \tag{34}$$

and the large frequency asymptotics follow from the small-t expansion (32) of the correlation function using the method of steepest descents [17]:

$$S_v(f) \sim \frac{1}{2\tau_0^2}\frac{1}{f} \quad \text{as } f\to\infty. \tag{35}$$

Note this is again independent of the details of the spectrum $S(f)$ of x and y, except for the dependence on the parameter τ_0, which may also be written as

$$\tau_0 = \left[\int_{-\infty}^{\infty} f^2 S(f)df\right]^{-1/2}. \tag{36}$$

This time scale can be used to estimate the low-frequency cutoff for the $1/f$ spectrum of (35). For frequencies significantly below $1/\tau_0$, the spectrum does not exhibit a $1/f$ scaling, as required by the fact that the process $v(t)$ is stationary; see the discussion following Eq. (11).

The appearance of the $1/f$ spectrum for a variety of correlation functions of x and y is shown in the numerical simulations of Sec. IV. The lack of a high-frequency cutoff in the $1/f$ spectrum (35) also confirms the nonexistence of the variance of v—the variance equals the integral over the spectrum, and the $1/f$ spectral tail means such an integration cannot yield a finite value. Indeed it is relatively straightforward to find the PDF $P(v)$ of the phase speed. This has been derived for the velocity following a contour in a random field in [18]; this random field formulation is immediately applicable to the stock market model of [14]. As explicitly demonstrated in [14], the PDF is

$$P(v) = \frac{\tau_0}{2[1+\tau_0^2 v^2]^{3/2}}, \tag{37}$$

which is symmetric in v, and so has mean zero. Note that the tails of $P(v)$ decay as $|v|^{-3}$ for large $|v|$, so the variance of v does not exist, in agreement with the results above.

IV. NUMERICAL SIMULATIONS

In order to check the theoretical predictions of Sec. III, and to extend them to cases where the correlation functions of $x(t)$ and $y(t)$ are not identical, we simulate fluctuation about a stable equilibrium point. The Jacobian matrix at the origin is assumed to have a simple form, so that the equations of motion for the x and y coordinates of the trajectory are

$$x' = -\alpha x + \eta(t),$$

$$y' = -\beta y + \xi(t), \tag{38}$$

with $\eta(t)$ and $\xi(t)$ being independent Gaussian colored noises. The noise processes may be generated in each real-

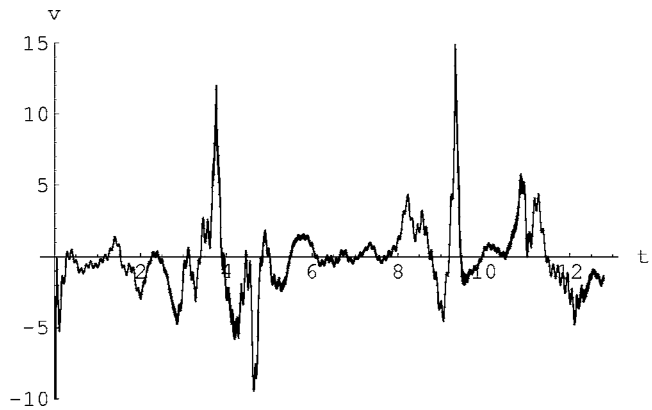

FIG. 1. Sample time series of $v(t)$ (units rad/sec), for parameters $\alpha=\beta=1/\tau$ and Ornstein-Uhlenbeck noise sources with correlation time τ. The time t is measured in units of τ.

ization using a finite sum of random modes [19,20], e.g.,

$$\eta(t) = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} a_n\cos(f_n t) + b_n\sin(f_n t), \qquad (39)$$

for large N (we take $N=100$). This spectral method can be used to generate Gaussian colored noise with a wide variety of different spectra and has been used to study dispersion in turbulent fluids [21,22]. Each of the a_n and b_n coefficients is chosen from a Gaussian distribution of random numbers, with mean zero and variance γ. The frequencies f_n are chosen from a random number distribution shaped as the spectrum $S_\eta(f)$ of the desired process. For instance, choosing the f_n values from a Cauchy distribution, which has PDF

$$P(f_n) = \frac{\tau}{\pi(1+\tau^2 f_n^2)}, \qquad (40)$$

gives a process $\eta(t)$ with correlation function (14). The correlation time τ of the noise sources is used to nondimensionalize all times and frequencies in the figures.

In each realization, the independent noise processes η and ξ are generated as in (39). Equations (38) for $x(t)$ and $y(t)$ are integrated exactly, and the phase speed

$$v(t) = \frac{xy' - yx'}{x^2+y^2} = \frac{x\xi - y\eta + (\alpha-\beta)xy}{x^2+y^2} \qquad (41)$$

is found at each moment of time. A short sample (up to $t=13\tau$) of a single realization of $v(t)$ is shown in Fig. 1. Standard fast Fourier transform (FFT) and periodogram methods yield the numerical spectrum of v; the average spectrum over 100 realizations is shown in Figs. 2 and 3.

In Fig. 2 the parameter values α and β in Eqs. (38) are taken to be equal. The correlation functions of $x(t)$ and of $y(t)$ are therefore identical, and can be found from the spectral form of (13) to be

$$R(t) = \frac{1}{1-\alpha\tau}\left[e^{-\alpha|t|} - \alpha\tau e^{-\frac{|t|}{\tau}}\right]. \qquad (42)$$

(after normalizing to unit variance). This is turn yields $R_v(t)$ from Eq. (31), and the spectrum $S_v(t)$ may then be found by

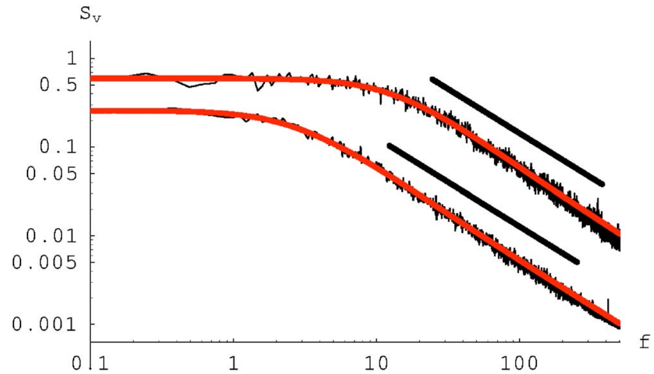

FIG. 2. (Color online) The spectrum (arbitrary units) of the phase speed $v(t)$ for noisy motion near an equilibrium point, as defined in Eqs. (38). The frequency f is in units of τ^{-1}. The numerical spectrum is shown in black; the theoretical spectrum resulting from Eq. (31) (gray line) is almost indistinguishable. Both cases use Ornstein-Uhlenbeck noise sources: lower curves correspond to parameter values of $\alpha=\beta=1/\tau$, upper curves have $\alpha=\beta=10/\tau$. Slopes corresponding to f^{-1} are shown as line segments. The numerical spectra are constructed from FFT samples of length 2^{14}, with time step $3.125\times10^{-3}\tau$, and averaged over 100 realizations.

numerical integration. This spectrum is shown with thick gray lines in Fig. 2. The lower curve in the figure corresponds to the case $\alpha=\beta=1/\tau$, while in the upper example the values are $\alpha=\beta=10/\tau$. In each case the theoretical result fits the numerical simulations very well, and a clear $1/f$ regime can be seen.

The time scale τ_0 defined in Eq. (33) is readily found from (42) to be

$$\tau_0 = \sqrt{\frac{\tau}{\alpha}} \qquad (43)$$

for the Ornstein-Uhlenbeck noise sources. As noted previously, τ_0^{-1} provides an estimate for the low-frequency cutoff of the $1/f$ spectrum—in particular, the dependence of the low-frequency cutoff on the tenfold increase in α between

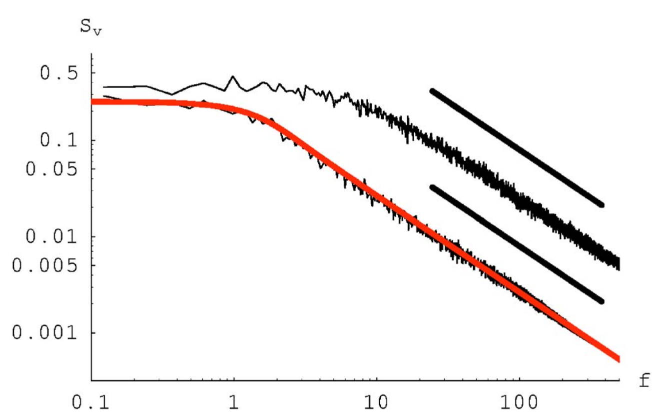

FIG. 3. (Color online) Spectrum (arbitrary units) of the phase speed $v(t)$; the frequency f is in units of τ^{-1}. Upper curve: numerical spectrum as in Fig. 2, but with $\alpha=1/\tau, \beta=10/\tau$. Lower curves: Numerical and theoretical spectrum for $\alpha=\beta=1/\tau$, with band-limited noise source spectrum given by Eq. (44).

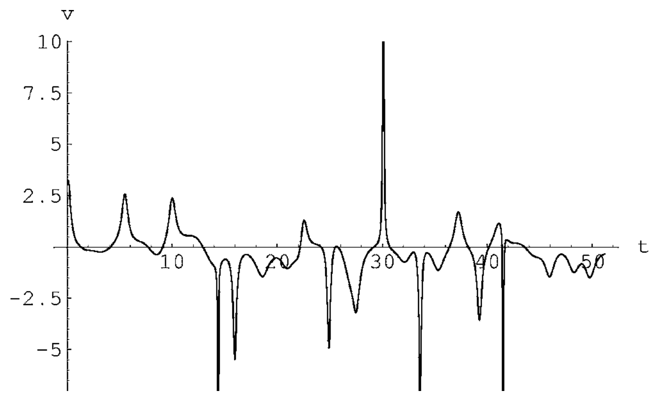

FIG. 4. Sample time series of $v(t)$ (units rad/sec), for parameters $\alpha=\beta=1/\tau$ with band-limited noise source spectrum given by Eq. (44). The time t is measured in units of τ.

lower and upper curves in Fig. 2 is clear. We note that the cutoff of the $1/f$ scaling can be moved to arbitrarily low frequencies by increasing the noise correlation time τ and/or by decreasing the Jacobian eigenvalue magnitude α.

One of the main results of the theoretical analysis is that the $1/f$ spectrum does not depend strongly on the details of the driving noise process. While the previous examples used noise with a power-law spectrum (15) at high frequencies, a more rapidly decaying spectrum will also yield a $1/f$ phase speed. This is demonstrated by the lower curve of Fig. 3, wherein the spectrum of the driving noise processes $\eta(t)$ and $\xi(t)$ is given by

$$S_\eta(f) = \frac{\gamma\tau}{\sqrt{2\pi}} e^{-\tau^2 f^2/2}. \tag{44}$$

Such an exponentially decaying spectrum is created by choosing the frequencies f_n in (39) from a Gaussian distribution of variance τ^{-2}. Taking parameters $\alpha=\beta=1/\tau$, the correlation function $R(t)$ of x and y, and therefore the phase speed correlation function $R_v(t)$, may be found exactly; the theoretically predicted spectrum is seen in Fig. 3 to closely match the numerical simulation results and to again give a $1/f$ spectrum. A sample time series of $v(t)$ is presented in Fig. 4.

The fundamental theoretical result (31) of our analysis is reliant upon the assumption that the correlation functions of $x(t)$ and $y(t)$ are identical. It is clear, however, that if $\beta \neq \alpha$ in (38), then the respective correlation functions (and variances) of x and y are no longer the same. We have not succeeded in extending the theoretical results of Sec. III to such cases, but present in Fig. 3 (upper curve) a typical result from our numerical simulations. Here the correlation function of the noise processes is (14), but the parameters governing $x(t)$ and $y(t)$ are substantially different: $\alpha=1/\tau, \beta=10/\tau$. Despite the resulting difference in the respective correlation functions, a $1/f$ tail in the spectrum is still clearly evident. This result leads us to conjecture that the theoretical result of Sec. III may have significantly wider applicability; moreover, it motivates current work on the general case of Eq. (4) with *arbitrary* Jacobian matrix J. This work will appear in a future publication.

V. CONCLUSIONS

We have introduced and analyzed a model of a continuous, stationary $1/f$ noise process

$$v(t) = \frac{d}{dt}\tan^{-1}\frac{y(t)}{x(t)}, \tag{45}$$

where $x(t)$ and $y(t)$ are Gaussian colored-noise processes with spectra obeying constraint (16). Our aim is not to produce a universal model explaining all $1/f$ processes in diverse experiments, but rather to develop a rare case where exact analytical results may be obtained. Several examples where $1/f$ processes of this type may arise are given; we adopt the example of noisy motion near a symmetric equilibrium point in two-dimensional dynamics, Eqs. (4) and (5). Our main result is the exact formula (31) giving the correlation function of $v(t)$ in terms of the correlation function of $x(t)$ and $y(t)$. Analysis of the small-t behavior of the correlation function proves that $v(t)$ has a $1/f$ spectrum with no high-frequency cutoff. Moreover, the $1/f$ spectrum of $v(t)$ is independent of the detailed form of the spectrum of $x(t)$ and $y(t)$, provided the latter has a finite second moment (36). Numerical simulations of noisy motion near an equilibrium point confirm these theoretical results. The lack of a high-frequency cutoff in the $1/f$ spectrum of $v(t)$ means that its variance is infinite, and thus it is necessarily a non-Gaussian process. This is confirmed by the explicit form of its one-time probability distribution function, Eq. (37).

Numerical simulations indicate that $1/f$ spectra are generated even when the equilibrium point is not symmetric, though theoretical results are not available in this case. This indicates that the results presented here are rather robust, and potentially extendable to a wider class of applications. A complete examination of the parameter regimes of Eq. (4) that generate $1/f$ noise is under way.

ACKNOWLEDGMENTS

This work is supported by funding from Science Foundation Ireland Investigator Award No. 02/IN.1/IM062. The author acknowledges the motivational questions on $1/f$ noise originally posed by Colin Lyden of Analog Devices.

[1] W. H. Press, Comments. Astrophys. **7**, 103 (1978).
[2] W. Li, www.nslij-genetics.org/wli/1fnoise
[3] M. B. Weissman, Rev. Mod. Phys. **60**, 537 (1988).
[4] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[5] J. Davidsen and M. Paczuski, Phys. Rev. E **66**, 050101 (2002).
[6] J. Davidsen and H. G. Schuster, Phys. Rev. E **62**, 6111 (2000).
[7] E. Milotti, Phys. Rev. E **51**, 3087 (1995).
[8] B. Kaulakys, Phys. Lett. A **257**, 37 (1999).
[9] K. L. Schick amd A. A. Verveen, Nature (London) **251**, 599 (1974).
[10] A. V. Oppenheim, Proc. IEEE **69**, 529 (1981).
[11] F. Casagrande *et al.*, Phys. Rev. Lett. **90**, 183601 (2003).
[12] A. Misra and D. I. Pullin, Phys. Fluids **9**, 2443 (1997).
[13] S. S. Girimaji and S. B. Pope, J. Fluid Mech. **220**, 427 (1990).
[14] J. P. Gleeson, Physica A **351**, 523 (2005).
[15] J. P. Gleeson, Phys. Rev. E **65**, 037103 (2002).
[16] H. Risken, *The Fokker-Planck Equation*, 2nd ed. (Springer, Berlin, 1989).
[17] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (Springer, New York, 1999), pp. 280–282.
[18] M. S. Longuet-Higgins, Proc. R. Soc. London, Ser. A **250**, 157 (1957).
[19] S. M. Prigarin, *Spectral Models of Random Fields in Monte Carlo Methods* (VSP, Utrecht, 2001).
[20] A. J. Majda and P. R. Kramer, Phys. Rep. **314**, 237 (1999).
[21] R. H. Kraichnan, Phys. Fluids **13**, 22 (1970).
[22] J. P. Gleeson, Phys. Fluids **12**, 1472 (2000).